TRANSFORMATIONS ACROSS THE FUNCTIONS

A Small Trip Down Memory Lane…

Recall the following types of transformations that were considered in Grade 11:

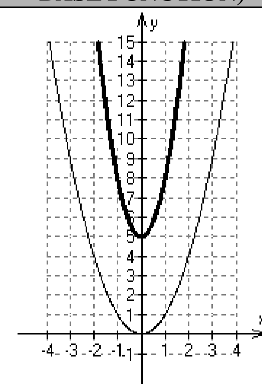
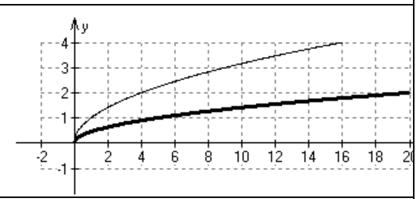
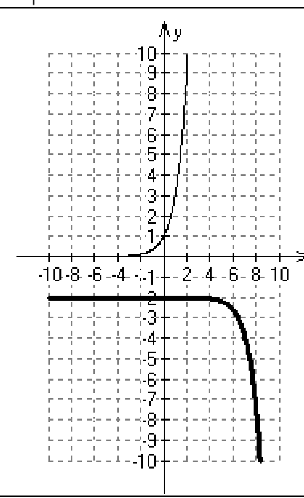
- vertical and horizontal translations (shifts)
- vertical and horizontal stretches and compressions
- vertical reflections in the x-axis and horizontal reflections in the y-axis

In particular, our study of the transformations above applied mainly to the following functions:

- $y = x$ (linear)
- $y = x^2$ (quadratic)
- $y = \sqrt{x}$ (square root)
- $y = \frac{1}{x}$
- $y = b^x$ (exponential)
- $y = \sin x$ (sine)
- $y = \cos x$ (cosine)

Examples

The purpose of the following table is to refresh your memory about some of the transformations studied in Grade 11.

FUNCTION	CORRESPONDING BASE FUNCTION	TRANSFORMATIONS APPLIED TO THE BASE FUNCTION	GRAPH (THIN GRAPH IS THE BASE FUNCTION)
$y = 3x^2 + 5$	$y = x^2$	• vertical stretch of factor 3 • vertical translation up 5 units	
$y = \sqrt{\frac{1}{5}x}$	$y = \sqrt{x}$	• horizontal stretch of factor 5	
$y = -\frac{1}{5}(3)^{x-5} - 2$	$y = 3^x$	• vertical reflection (in the x-axis) • vertical compression of factor $\frac{1}{5}$ • horizontal translation right 5 units • vertical translation down 2 units	

Some more functions for discussion…

- $y = 3x + 7$
- $y = \dfrac{1}{x-7} + 5$
- $y = -4(2x+8)^2 - 15$
- $y = 5^{-\frac{1}{2}x+3}$
- $y = -4\sqrt{x-7}$
- $y = \dfrac{2}{5}\cos(3x - 180°) - 4$

<u>Making the Jump to Generalization</u>

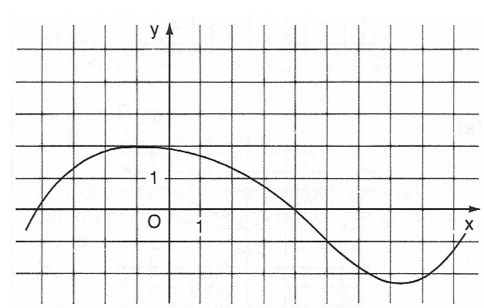
Sometimes we may need to work with functions whose equations are either unknown to us or are very complex. When working with these functions and their graphs, however, we can easily use general function notation to describe transformations.

What does this mean? Well, think of the graph of any function you want, even if it's a wild and crazy graph whose equation is completely unknown to you! Let's represent this function by $f(x)$, which just means that our function *f* takes *x* as its input and gives $f(x)$ as its output. Now, how could we represent $f(x)$ after its graph has been vertically stretched by a factor 3 and shifted up 5 units? No problem… $3f(x)+5$. Think about what we're saying when we write $3f(x)+5$: for each input value *x*, we are multiplying the output value $f(x)$ (which can be thought of as *y* on the graph) by 3 and then adding 5. The graph would thus be vertically stretched by a factor of 3 and then shifted up 5 units. This idea shouldn't seem new to you, but here are a few more examples anyway:

FUNCTION	TRANSFORMATIONS APPLIED TO THE GRAPH OF $y = f(x)$
$y = f(5x)$	• horizontal compression of factor $\frac{1}{5}$
$y = -f(x-2)+3$	• vertical reflection (in the *x*-axis) • horizontal translation right 2 units • vertical translation up 3 units
$y = f\left(-\frac{1}{3}x - 2\right)$ Factor: $y = f\left[-\frac{1}{3}(x+6)\right]$	• horizontal reflection (in the *y*-axis) • horizontal stretch of factor 3 • horizontal translation left 6 units

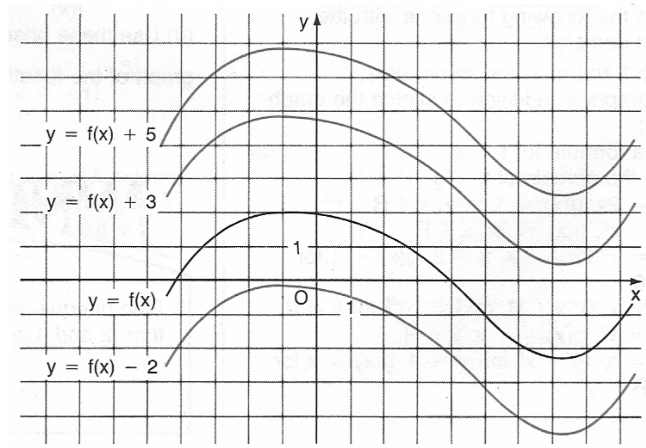
Note: Generalization is convenient because it allows us to easily describe transformations on any function, regardless of how complicated the function may be. It also gives us a way to discuss transformations without referring specific functions.

Example
Given the graph of $y = f(x)$ to the right**,** sketch the graphs of $y = f(x)+3$, $y = f(x)+5$ and $y = f(x)-2$.



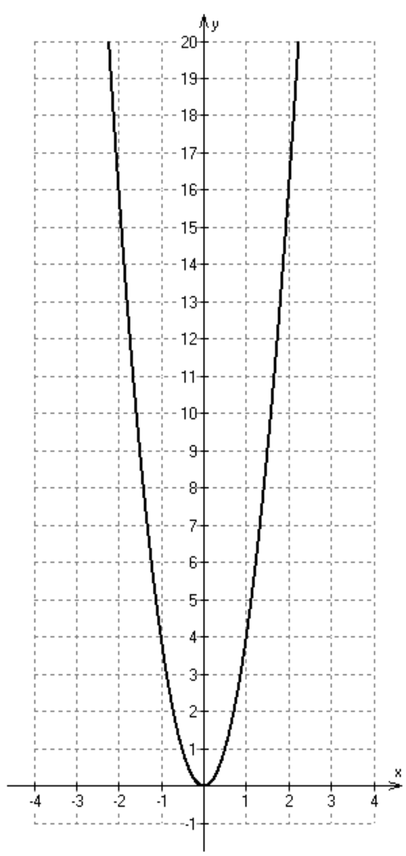
Solution

- $y = f(x)+3$ is the graph of $y = f(x)$ shifted up 3 units.
- $y = f(x)+5$ is the graph of $y = f(x)$ shifted up 5 units.
- $y = f(x)-2$ is the graph of $y = f(x)$ shifted down 2 units.



A little more about generalization…

Consider the function $g(x) = f(x)+2$ (which immediately seems strange and perhaps somewhat confusing!). Now, suppose someone kindly asked you to sketch a graph of this function. Well, you find yourself in quite a pickle! Why? The problem is that you don't know anything about $f(x)$. All you know at this point is that the function g adds 2 to $f(x)$, thus shifting the graph of $f(x)$ up 2 units. If $f(x)$ represents a line, then $g(x)$ would represent that line shifted up 2 units. If $f(x)$ represents a parabola, then $g(x)$ would represent that parabola shifted up 2 units. The statement $g(x) = f(x)+2$ is a generalization, as it may be used to refer to any function $f(x)$ that has been shifted up 2 units.



An Interesting Question for Discussion

On a math exam, students were asked to give an equation for the graph on the left.

Matt stated that this graph is the graph of $y = x^2$ vertically stretched by a factor of 4 and concluded that its equation is $y = 4x^2$.

Becky claimed that this graph is the graph of $y = x^2$ horizontally compressed by a factor of $\frac{1}{2}$ and concluded that its equation is $y = (2x)^2$.

Who was correct? Explain. Can you find similar examples of this situation?