SOLUTIONS

UNIT 3 – DERIVATIVES AND THEIR APPLICATIONS

SUPPLEMENTARY REVIEW PROBLEMS

1) The position of a particle is given by $s(t) = t^2 - 8t + 4,\ t \geq 0$, where s is measured in metres and t in seconds. Find all values of t for which the particle moves in the negative direction.

$v(t) = s'(t) = 2t - 8$

When particle is moving in the negative direction, its velocity is negative.

$2t - 8 < 0$

$2t < 8$

$t < 4$

Therefore, the object is moving in the negative direction for $0 \leq t < 4$.

2) A water balloon is thrown straight upward. Its height as a function of time can be modeled by $h(t) = 13t - 4.9t^2$, where the height is measured in metres and the time is in measures in seconds.

a) What is the maximum height of the water balloon?

$h'(t) = 13 - 9.8t$

For maximum height, $h'(t) = 0$ (tangent slope is zero).

$0 = 13 - 9.8t$

$9.8t = 13$

$t = \dfrac{13}{9.8}$

$t = 1.33$

Therefore,

$h(1.33) = 13(1.33) - 4.9(1.33)^2$

$h(1.33) = 8.62$

Therefore, the maximum height is 8.62 metres.

b) How long does the thrower have to get out of the way before the water balloon hits the ground and pops?

When balloon hits ground, $h(t) = 0$.

$$0 = 13t - 4.9t^2$$
$$0 = t(13 - 4.9t)$$
$$\therefore t = 0 \text{ or } 13 - 4.9t = 0$$
$$t = 0 \quad \text{or} \quad -4.9t = -13$$
$$t = 0 \quad \text{or} \quad t = \frac{-13}{-4.9}$$
$$t = 0 \quad \text{or} \quad t = 2.65$$

The thrower has 2.65 seconds to get out of the way.

3) The position, in kilometers, of a particle at t hours is given by $d(t) = t^3 - 15t^2 + 44t + 60$, where $t \geq 0$.

a) What is the initial position of the particle?

$$d(0) = 60$$

Therefore, the initial position is 60 km.

b) What is the particle's velocity at 4 hours?

$$d'(t) = 3t^2 - 30t + 44$$
$$d'(4) = 3(4)^2 - 30(4) + 44$$
$$d'(4) = 48 - 120 + 44$$
$$d'(4) = -28$$

Therefore, the velocity at 4 hours is -28 km/h.

c) What is the total distance that the particle travels in the first 10 hours?

Find the times when the object is stopped by setting $d'(t) = 0$:

$$3t^2 - 30t + 44 = 0$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{30 \pm \sqrt{(-30)^2 - 4(3)(44)}}{2(3)}$$

$$t = \frac{30 \pm \sqrt{900 - 528}}{6}$$

$$t = \frac{30 \pm \sqrt{372}}{6}$$

$$t = \frac{30 \pm 19.3}{6}$$

$$t = \frac{49.3}{6} \quad \text{or} \quad t = \frac{10.7}{6}$$

$$t = 8.2 \quad \text{or} \quad t = 1.8$$

Find the corresponding positions for all of the relevant points:

t	*d(t)*
0	60
1.8	96.4
8.2	-36.4
10	0

The distance traveled between $t = 0$ and $t = 1.8$ is 36.4 kilometres.
The distance traveled between $t = 1.8$ and $t = 8.2$ is 132.8 kilometres.
The distance traveled between $t = 0$ and $t = 1.8$ is 36.4 kilometres.

The total distance traveled is 205.6 km.

4) A health club has 1500 members who are charged \$60 each per year. The club proposes to increase the annual charges. For every increase of \$10, the club will lose 125 members. What increase will bring the maximum revenue to the club?

Let x represent the number of \$10 increases.

Revenue = price × # sold

$$R = (60+10x)(1500-125x)$$
$$R' = (10)(1500-125x)+(-125)(60+10x)$$
$$= 15000-1250x-7500-1250x$$
$$= 7500-2500x$$

For critical numbers,

$$0 = 7500-2500x$$
$$x = \frac{7500}{2500}$$
$$x = 3$$

Check for maximum:

$$R'' = -2500$$
$$< 0 \quad \text{Therefore, maximum at } x = 3$$

Therefore, 3 price increases of \$10 will maximize revenue. In other words, a \$30 increase will bring the maximum revenue to the club.

5) Determine the maximum slope of a tangent to the curve $y = -x^3 + 3x^2 + 9x - 27$.

$$y = -x^3 + 3x^2 + 9x - 27$$
$$y' = -3x^2 + 6x + 9$$

For maximum slope of tangent, we need to maximize y'.

$$y'' = -6x + 6$$

For max/min,

$$0 = -6x + 6$$
$$x = 1$$

	$x < 1$	$x > 1$
Sign of $y'' = -6x + 6$	+	-
Increase/Decrease for y'	↗	↘

Therefore, maximum tangent slope occurs when $x = 1$.

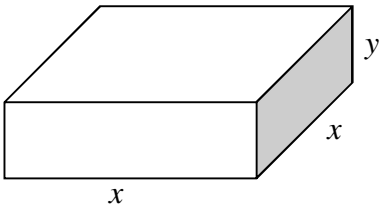
When $x = 1$,

$$y' = -3(1)^2 + 6(1) + 9$$
$$= 12$$

Therefore, maximum tangent slope is 12.

6) A closed rectangular box with a square base is to be made such that its volume is 1000 m^3. The cost of material per square metre for the bottom is \$15, for the top is \$25, and for the sides is \$20. Find the dimensions of the box such that the cost is minimized.

Let x and y represent the side lengths as shown in the diagram.



$$V = lwh$$
$$1000 = xxy$$
$$1000 = x^2 y$$
$$y = \frac{1000}{x^2}$$

$$C = 15x^2 + 25x^2 + 20(4)xy$$
$$C = 40x^2 + 80xy$$
$$C = 40x^2 + 80x\left(\frac{1000}{x^2}\right)$$
$$C = 40x^2 + 80000x^{-1}$$
$$C' = 80x - \frac{80000}{x^2}$$

For critical numbers,

$$0 = 80x - \frac{80000}{x^2}$$
$$\frac{80000}{x^2} = 80x$$
$$x^3 = \frac{80000}{80}$$
$$x = 10$$

Check for minimum:

$$C'' = 80 + \frac{160000}{x^3}$$

When $x = 10$,

$$C'' = 80 + \frac{160000}{10^3}$$
$$C'' = 240$$
$$C'' > 0$$

Therefore, minimum occurs when $x = 10$

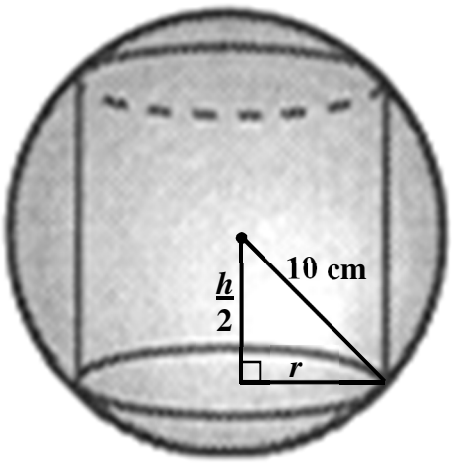
When $x = 10$,

$$y = \frac{1000}{10^2}$$
$$= 10$$

Therefore, to minimize cost, all sides must have a length of 10 m.

7) A cylinder is inscribed in a sphere that has a diameter of 20 cm. What are the dimensions of the cylinder if its volume is maximized?

Let V, h and r be the volume, height and radius of the cylinder, respectively.



$$10^2 = \left(\frac{h}{2}\right)^2 + r^2$$

$$r^2 = 100 - \frac{h^2}{4}$$

$$V = \pi r^2 h$$

$$= \pi\left(100 - \frac{h^2}{4}\right)h$$

$$= 100\pi h - \frac{\pi h^3}{4}$$

$$V' = 100\pi - \frac{3\pi h^2}{4}$$

For critical numbers,

$$0 = 100\pi - \frac{3\pi h^2}{4}$$

$$\frac{3\pi h^2}{4} = 100\pi$$

$$h^2 = \frac{400}{3}$$

$$h \doteq \pm 11.55$$

(negative answer is inadmissible)

Check for maximum:

$$V'' = -\frac{3\pi h}{2}$$

When $h = 11.55$, $V'' < 0$.

Therefore, maximum occurs when $h = 11.55$

To find r:

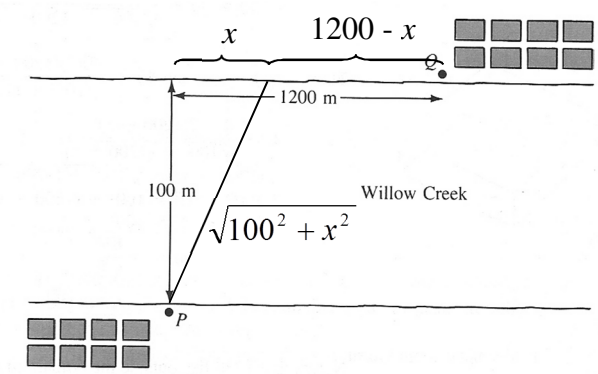
$$r^2 = 100 - \frac{h^2}{4}$$

$$r = \sqrt{100 - \frac{11.55^2}{4}}$$

$$r \doteq 8.16$$

Therefore, if its volume is maximized, the cylinder has a height of 11.55 cm and a radius of 8.16 cm.

8) A cable television company is laying cable in an area with underground utilities. Two subdivisions are located on opposite sides of Willow Creek, which is 100 m wide. The company must connect points P and Q with cable, where Q is on the north bank 1200 m east of P, as shown below. It costs \$40/m to lay cable underground and \$80/m to lay cable underwater. What is the least expensive way to lay the cable?



Let x be the distance shown on the diagram.

$$C = 80\sqrt{100^2 + x^2} + 40(1200 - x)$$

$$= 80\left(10000 + x^2\right)^{\frac{1}{2}} + 48000 - 40x$$

$$C' = 40\left(10000 + x^2\right)^{-\frac{1}{2}}(2x) - 40$$

For critical numbers,

$$0 = 40\left(10000 + x^2\right)^{-\frac{1}{2}}(2x) - 40$$

$$40 = \frac{80x}{\sqrt{10000 + x^2}}$$

$$40\sqrt{10000 + x^2} = 80x$$

$$\sqrt{10000 + x^2} = 2x$$

$$10000 + x^2 = 4x^2$$

$$10000 = 3x^2$$

$$x = \pm\sqrt{\frac{10000}{3}} \quad \text{(negative answer is inadmissible)}$$

$$x \doteq 57.74$$

Check for minimum:

	$x < 57.74$	$x > 57.74$
Sign of C'	-	+
Increase/Decrease of C	↘	↗

Therefore, we have a minimum when $x = 57.74$.

$1200 - 57.74 = 1142.26$

Therefore, to minimize cost, the cable should be run underwater to the point on the north bank that is 1142.26 m from point Q.

9) Find the points on the parabola $y = 6 - x^2$ that are closest to the point (0, 3). Express your answer in **exact form**.

We need to minimize the distance between the point (0, 3) and the point (x, $6 - x^2$).

$$D = \sqrt{\left(6 - x^2 - 3\right)^2 + (x - 0)^2}$$
$$= \sqrt{\left(3 - x^2\right)^2 + x^2}$$

To minimize D, we can minimize the expression under the square root.

Let $f = \left(3 - x^2\right)^2 + x^2$

$$f' = 2\left(3 - x^2\right)(-2x) + 2x$$
$$= -2x\left[2\left(3 - x^2\right) - 1\right]$$
$$= -2x\left(5 - 2x^2\right)$$

$x = 0$ or $x = \pm\sqrt{2.5}$

	$x < -\sqrt{2.5}$	$-\sqrt{2.5} < x < 0$	$0 < x < \sqrt{2.5}$	$x > \sqrt{2.5}$
Sign of f'	-	+	-	+
Increase/Decrease of f	↘	↗	↘	↗

Therefore, we have a minimum when $x = -\sqrt{2.5}$ and when $x = \sqrt{2.5}$. (We see from the equation for D that these two values will give the same distance, since squaring $-\sqrt{2.5}$ gives the same result as squaring $\sqrt{2.5}$.)

Therefore, the points on the parabola that are closest to (0, 3) are $\left(-\sqrt{2.5}, 3.5\right)$ and $\left(\sqrt{2.5}, 3.5\right)$.