THE LIMIT OF A FUNCTION

THE LIMIT OF A FUNCTION

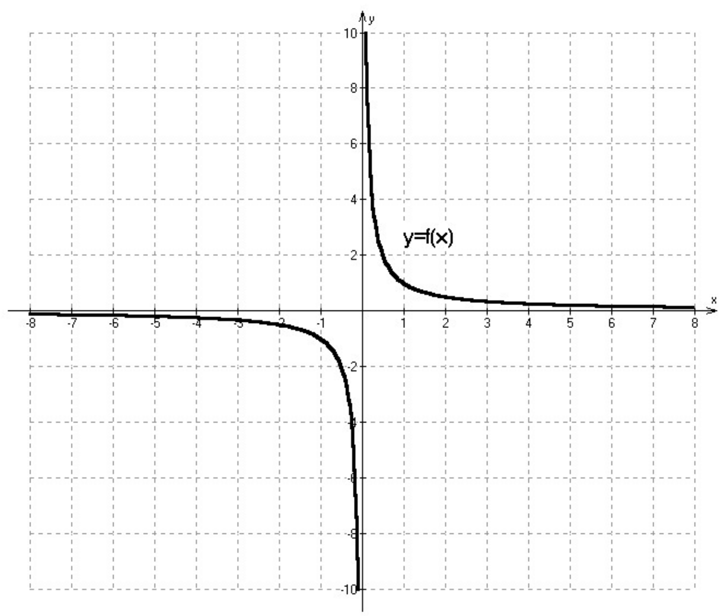
The notation $\lim_{x \to a} f(x) = L$ is read, "the limit of $f(x)$, as x approaches *a,* is L." It means that the value of the function $f(x)$ approaches the number L $(L \in \Re)$ as x approaches a from both sides.

Note that this definition is very specific. To obtain the limit, we need to approach the *a* value **from both sides**.

To see why it is so important to approach the *a* value from both sides when finding a limit, consider the following example:

Example

Find $\lim_{x \to 0} f(x)$ for the function shown in the graph below.



As $x \to 0^+$ (read "as *x* approaches 0 from the positive side"), the function approaches $+\infty$.
As $x \to 0^-$ (read "as *x* approaches 0 from the negative side"), the function approaches $-\infty$.
Clearly, there is no single "right answer" here and we say, "the limit does not exist".

Using limit notation, we write

$$\lim_{x \to 0^+} f(x) = +\infty \qquad \lim_{x \to 0^-} f(x) = -\infty$$

$\lim_{x \to 0} f(x)$ does not exist, since the previous two **one sided limits** did not give the same result.

Formally,

ONE-SIDED LIMITS

$\lim_{x\to a^-} f(x)$ denotes the limit of $f(x)$ as x approaches a from the **left side**.

$\lim_{x\to a^+} f(x)$ denotes the limit of $f(x)$ as x approaches a from the **right side**.

TWO-SIDED LIMIT

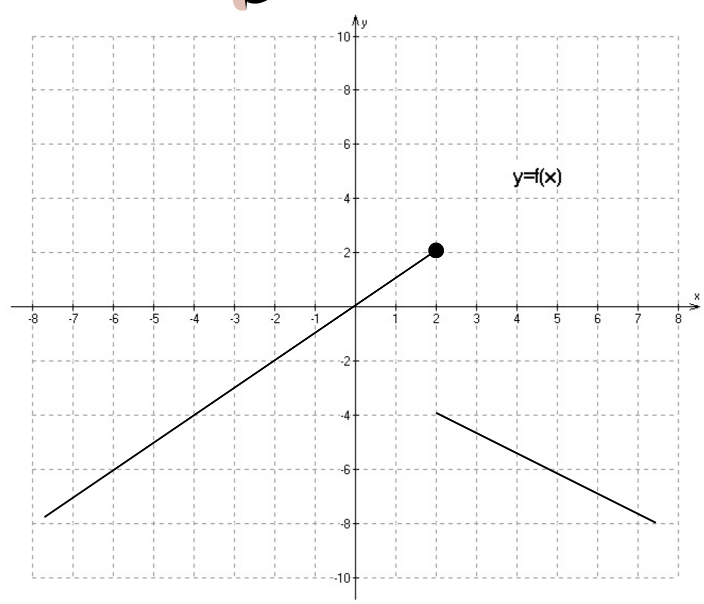
If $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$, then $\lim_{x\to a} f(x)$ exists and is equal to L.

$\lim_{x\to a} f(x)$ is called a **two sided limit.**

LIMITS THAT FAIL TO EXIST

If $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$, then $\lim_{x\to a} f(x)$ does not exist.

Example
Determine the following limits for the function shown on the right.



$$\lim_{x\to 2^+} f(x) =$$

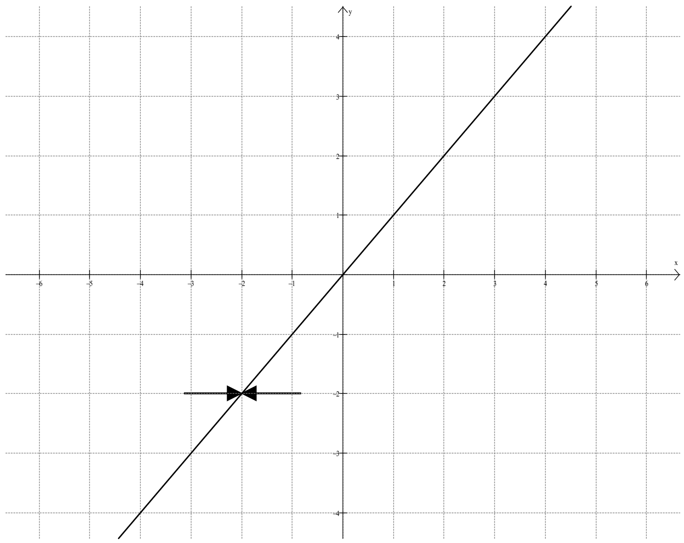
$$\lim_{x\to 2^-} f(x) =$$

$$\lim_{x\to 2} f(x) =$$

In the previous two examples, the limit did not exist due to a discontinuity in the graph. At an x-value where there is no discontinuity, however, finding the limit is very easy, as it is usually just the value of the function at that point. The following examples demonstrate this idea.

Example
Evaluate the following limits.

a) $\lim_{x \to -2} x$

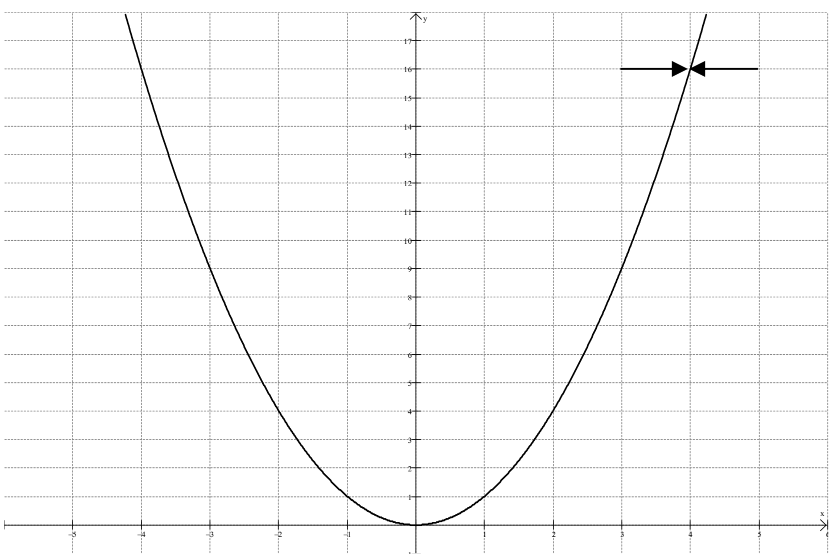


$$\lim_{x \to -2^+} f(x) =$$

$$\lim_{x \to -2^-} f(x) =$$

$$\lim_{x \to -2} f(x) =$$

b) $\lim_{x \to 4} x^2$



$$\lim_{x \to 4^+} f(x) =$$

$$\lim_{x \to 4^-} f(x) =$$

$$\lim_{x \to 4} f(x) =$$

As seen in the previous two examples, when working with continuous functions, there is often no difference between the limit approaching a certain x-value and the function's y-value at that point.

When the function is more complicated, we can still use this idea. **Just remember that the limit only exists when the left-side and right-side limits exist and are equal.**

Note: The terminology, "the limit does not exist" is often used to describe either of the following situations.
1) The limit cannot be evaluated.
2) The "limit" is ∞ or $-\infty$.