Polynomial Functions — An Introduction

Definition of a Polynomial in One Variable

A polynomial is an expression of the form

$$a_n x^{n} + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^{2} + a_1 x + a_0$$

where

1) n is a whole number.
   - all of the exponents are whole numbers (0, 1, 2, 3, etc.)

2) the coefficients $a_0, a_1, a_2, \ldots, a_n$ are real numbers.

<u>Question for Discussion</u>
"What's a real number?"

SOME EXAMPLES

These are polynomial expressions.	These are not polynomial expressions.
$3x^2 - 5x + 3$	$\sqrt{x} + 5x^3$
$-4x + 5x^7 - 3x^4 + 2$	$\frac{1}{2x+5}$
$\frac{2}{5}x^3 - 3x^5 + 4$	$6x^3 + 5x^2 - 3x + 2 + 4x^{-1}$
$\sqrt{4}x^3 - \frac{\sqrt{5}}{3}x^2 + 2x - \frac{1}{4}$	$\frac{3x^2 + 5x - 1}{2x^2 + x - 3}$
$3x - 5$	$4^x + 5$
-7	$\sin(x - 30)$
$-4x$	$x^2y + 3x - 4y^{-2}$
$(2x - 3)(x + 1)^2$	$3x^3 + 4x^{2.5}$

- A polynomial can use a variable other than x.
  Example: $2.4t^3 + 7t^2 - 5.5t + 1$
- A polynomial can contain more than one variable.
  Example: $3x^2 + 2t + 5x^3t^2$

SOME MORE TERMINOLOGY

Degree of a Polynomial
The degree of a polynomial is the value of the highest exponent on the variable.

Example: $7x^{3} - 12x^2 + 5x - 8$ Degree is <u>3</u>.

Leading Coefficient
The leading coefficient of a polynomial is the coefficient of the highest power of the variable.

Example: $7x^3 - 12x^2 + 5x - 8$ Leading coefficient is is <u>7</u>.

Polynomial Function
A polynomial function is a function for which the rule is a polynomial.

Example: $f(x) = 7x^3 - 12x^2 + 5x - 8$

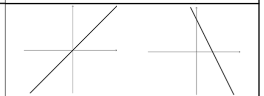
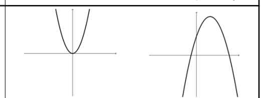
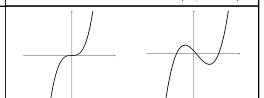
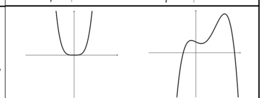
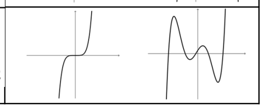
Name that degree and leading coefficient!

$-8x^4 - 2x^3 + x + 1$ — Degree: 4; Leading Coefficient: -8

$y = 5x^3 + 9x + x^4 - 3x^2 - 8$ — Degree: 4; Leading Coefficient: 1

$f(x) = 1.5x^3 + 0.8x^5 - 4.1x^7$ — Degree: 7; Leading Coefficient: -4.1

EXAMPLES OF POLYNOMIAL FUNCTIONS

DEGREE	COMMON NAME	EQUATION EXAMPLES	GRAPH EXAMPLES
0	Constant	$y=3$ $y=-5$	
1	Linear	$y=x$ $y=-2x+5$	
2	Quadratic	$y=x^2$ $y=-2x^2+5x+3$	
3	Cubic	$y=x^3$ $y=x^3-x^2-4x+1$	
4	Quartic	$y=x^4$ $y=-x^4+4x^3-3x+7$	
5	Quintic	$y=x^5$ $y=1.3x^5+x^4-10x^3-4.2x^2+10x+2$	

How high can we go with the degree of a polynomial function?

As high as we want, but…

- there aren't really common names for high-degree polynomials.
- we'll spend most of our time working with degree 5 or less.

I wonder how the coefficients affect the shape of the graph…

DESCRIBING POLYNOMIAL FUNCTIONS

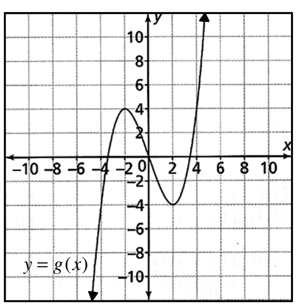
TURNING POINTS

- When a function switches from increasing to decreasing, or vice versa, we get a **turning point**.
- Turning points are higher or lower than all nearby points.
- Turning points are also known as **local maximum points** or **local minimum points**.

OTHER PROPERTIES FOR DESCRIBING POLYNOMIAL FUNCTIONS

- Domain and Range
- Zeros
- *y*-intercept
- Absolute extreme values
- Symmetry
- End behaviour

EXAMPLE:



Domain: $\{x \in \Re\}$

Range: $\{y \in \Re\}$

Zeros: $-3.5,\ 0,\ 3.5$

y-intercept: 0

Interval(s) of increase: $x<-2,\ x>2$

Interval(s) of decrease: $-2<x<2$

Absolute Maximum Value: None

Absolute Minimum Value: None

There is local maximum point at __−2__ with a local maximum value of __4__.

There is local minimum point at __2__ with a local minimum value of __−4__.

Symmetry: Odd, since $g(-x)=-g(x)$

End Behaviour: As $x\to\infty,\ y\to\infty$
As $x\to-\infty,\ y\to-\infty$

Note: We will look more closely at end behaviour, zeros and turning points in the next lesson.

FINITE DIFFERENCES

Recall that the **first differences** of a linear function are equal.

$y=5x-12$

x	y	First differences
0	-12	
1	-7	5
2	-2	5
3	3	5
4	8	5
5	13	5

Similarly, the **second differences** of a quadratic function are equal.

$y=3x^2-19x+12$

x	y	First differences	Second differences
0	12		
1	-4	-16	
2	-14	-10	6
3	-18	-4	6
4	-16	2	6
5	-8	8	6

What do you think will happen when the function is cubic?

The **third differences** will be equal!

In general, for a polynomial function of degree *n*, the n^{th} differences are equal.

$y=5x^3-7x^2+2x-4$

x	y	First differences	Second differences	Third differences
0	-4			
1	-4	0		
2	12	16	16	
3	74	62	46	30
4	212	138	76	30
5	456	244	106	30

As we continue our investigation of rates of change, the reason why this relationship exists will become apparent.