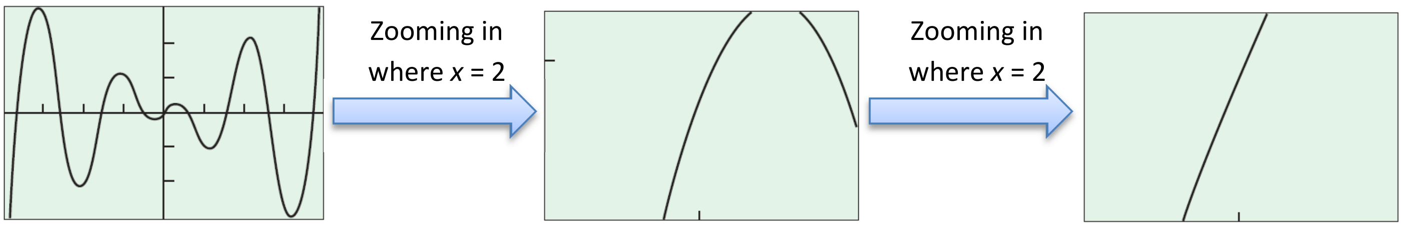
Local Linearity…A Little Review

Recall that when we refer to the "slope of a curve," we are actually referring to the slope of a tangent line. In fact, if we stay close enough to the point of tangency, the tangent line is actually a good representation of the curve itself!

For this reason, we say that differentiable curves are locally linear. Recall the following illustration of local linearity from an earlier lesson.

Imagine zooming in on a smooth curve at a point. If you keep zooming in, what would the graph start to look like?



Question for Discussion
What condition must be satisfied in order for a curve to be locally linear at a point?

Therefore, the tangent line can actually be used to approximate the curve as near the point of tangency, since the *y*-values are almost the same.

Now, we know that the equation of a line can be written in point-slope form as follows:

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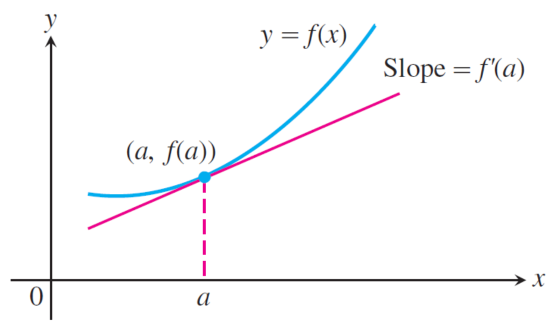
Consider the tangent line to a curve $f(x)$ at the point where $x = a$.

The point of tangency is $\Big(\quad , \quad \Big)$. Furthermore, we know that the slope of this tangent line is ___________.



Therefore, if we stay close to $x = a$, the curve can be approximated by the following line:

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DEFINITION Linearization

If f is differentiable at $x = a$, then the equation of the tangent line,

$$L(x) = f(a) + f'(a)(x - a),$$

defines the **linearization of f at a.** The approximation $f(x) \approx L(x)$ is the **standard linear approximation of f at a.** The point $x = a$ is the **center** of the approximation.

Examples

1) a) Determine the linearization of $f(x)=\sqrt{1+x}$ at $x = 0$.

b) Use your linearization to approximate the value of $\sqrt{1.02}$ without a calculator.

c) Use a calculator to determine the accuracy of the approximation.

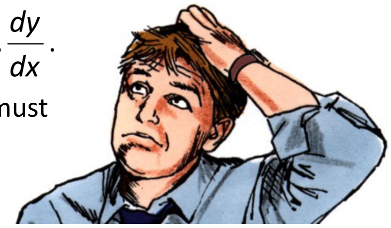
d) The results above can actually be extended to all powers of 1 + *x*. Specifically, for small values of *x* (close to 0), we get $(1+x)^k \approx 1+kx$. Use this property to find linearizations for the following functions for values of x close to 0.

i) $\sqrt[3]{1-x}$ ii) $\dfrac{1}{1-x}$ iii) $\sqrt{1+5x^4}$

e) Use linearization to approximate the value of $\sqrt{123}$.

Differentials…An Introduction Just Because You Should Know About Them!

The ideas behind linear approximations are sometimes discussed using the idea of **differentials**. To gain an understanding of differentials, recall the Leibniz notation for the derivative, $\dfrac{dy}{dx}$.



Although it is tempting to think of this notation as a quotient of real numbers, we must remember that it is actually a limit of quotients in which the numerator and denominator approach zero.

The variables *dx* and *dy* seen in Leibniz notation are called **differentials** and are **defined** as follows:

DEFINITION Differentials

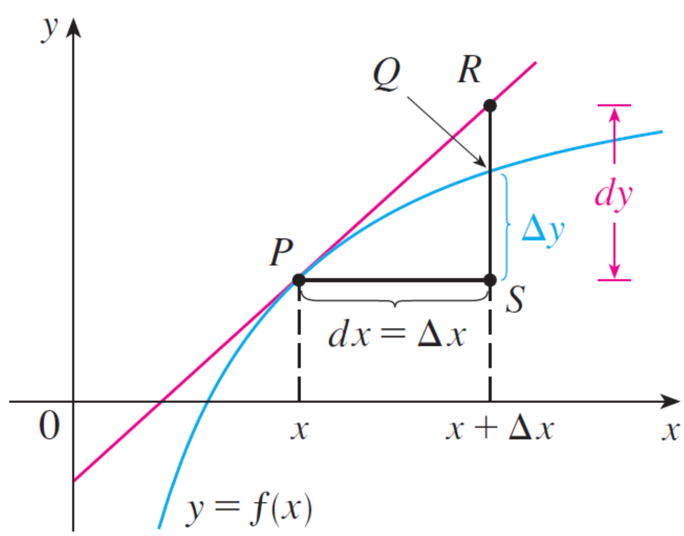
Let $y = f(x)$ be a differentiable function. The **differential** dx is an independent variable. The **differential** dy is

$$dy = f'(x)\,dx.$$

Notice that if $dx \neq 0$, we can divide both sides by *dx* to obtain $\dfrac{dy}{dx} = f'(x)$. This equation should look familiar, but now the left side can genuinely be interpreted as a ratio of differentials.

So, *dy* is a dependent variable. It depends on the values of *x* and *dx*. If *dx* is given a specific value and *x* is taken to be some specific number in the domain of *f*, then the numerical value of *dy* is determined.

To understanding the meaning of differentials geometrically, consider the diagram on the right.



Let $P(x, f(x))$ and $Q(x + \Delta x, f(x + \Delta x))$ be points on the graph of *f* and let $dx = \Delta x$. The corresponding change in *y* is

$$\Delta y = f(x + \Delta x) - f(x).$$

The slope of the tangent line PR is the derivative $f'(x)$. Thus the directed distance from S to R is $f'(x)dx = dy$. Therefore *dy* represents the amount that the tangent line rises or falls (the change in the linearization), whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when *x* changes by an amount *dx*.

Examples

1) Use the definition of differentials to find the differential *dy* and evaluate *dy* for the given values of *x* and *dx*. Discuss the geometric interpretation of your results.

a) $y = x^5 + 37x, \quad x = 1, \quad dx = 0.01$

b) $y = \sin 3x, \quad x = \pi, \quad dx = -0.02$

c) $x + y = xy, \quad x = 2, \quad dx = 0.05$

Why do differentiation rules also apply to differentials?

2) The radius of a circle increases from 10 m to 10.1 m. Use *dA* to estimate the increase in the circle's area *A*. Compare this estimate with the true change ΔA and find the approximation error.