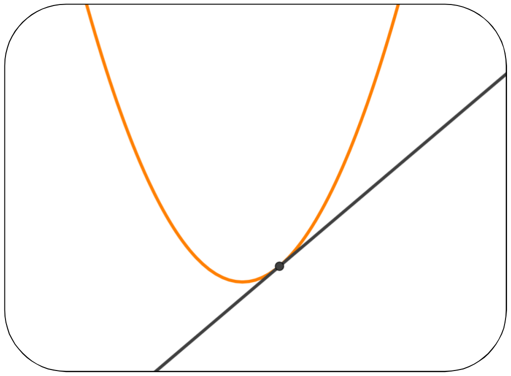


INSTANTANEOUS RATE OF CHANGE

Introductory Example

The height, in metres, of a model rocket t seconds after it is launched is given by $h(t) = -4.9t^2 + 25t + 2$.

At this point, we should be able to find the average velocity on any interval.

QUESTION FOR DISCUSSION
How could we determine the instantaneous velocity at 1 second?

Consider the following table:

Interval	Δh	Δt	Average Rate of Change $\left(\frac{\Delta h}{\Delta t}\right)$
$1 \le t \le 2$	$h(2) - h(1)$ $= 32.4 - 22.1$ $= 10.3$	2 – 1 = 1	$\frac{10.3}{1} = 10.3$ m/s
$1 \le t \le 1.5$	6.375	0.5	12.75 m/s
$1 \le t \le 1.1$	1.471	0.1	14.71 m/s
$1 \le t \le 1.01$	0.15151	0.01	15.151 m/s
$1 \le t \le 1.001$	0.0151951	0.001	15.1951 m/s

The height, in metres, of a model rocket t seconds after it is launched is given by $h(t) = -4.9t^2 + 25t + 2$.

Notice that as the time interval around $t = 1$ becomes very small, the average rate of change seems to approach a specific value.

Interval	Δh	Δt	Average Rate of Change $\left(\frac{\Delta h}{\Delta t}\right)$
$1 \le t \le 2$	10.3	1	10.3 m/s
$1 \le t \le 1.5$	6.375	0.5	12.75 m/s
$1 \le t \le 1.1$	1.471	0.1	14.71 m/s
$1 \le t \le 1.01$	0.15151	0.01	15.151 m/s
$1 \le t \le 1.001$	0.0151951	0.001	15.1951 m/s

Based on this trend, we can conclude that the instantaneous rate of change at $t = 1$ is approximately **15.2 m/s**.

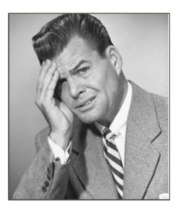
GETTING TO THE POINT

In the previous example, when we were finding the instantaneous rate of change, we were actually attempting to determine the slope of the function at a specific point (where $t = 1$).

WAIT A MINUTE!!!

Is it possible to find the slope at a single point? Don't we always need 2 points to calculate slope? How could we possibly determine the slope at a point on a curved graph?

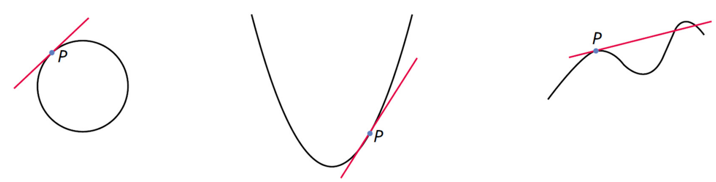
We need to discuss another concept now…



INTRODUCING… THE TANGENT

The instantaneous rate of change of a function at a given point is the slope of the **tangent line** at that point.

Consider the diagrams below. In each case, the **tangent line** "touches" or "rests on" a point P on the curve. Point P is called the point of tangency.



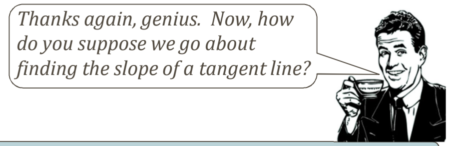
NOTES:
- The tangent line does not "pierce through" the curve at the point of tangency.
- The tangent line may also intersect the graph at points other than the point of tangency.

In the diagrams above, notice how the slope of the tangent line describes the curve's rate of change at the point of tangency.

That is, the steepness of the tangent line represents the steepness of the graph at the point of tangency.

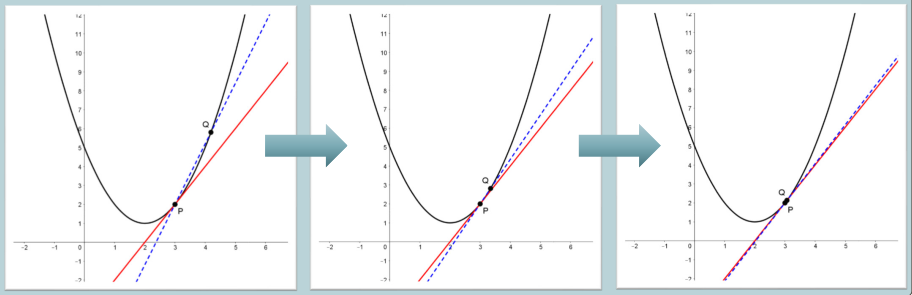
So, if we want to find the instantaneous rate of change at a particular point, we just need to find the slope of the tangent at that point!

We approximate the slope of a tangent line by using a series of secant lines (average rates of change), similar to what we did in the model rocket example.

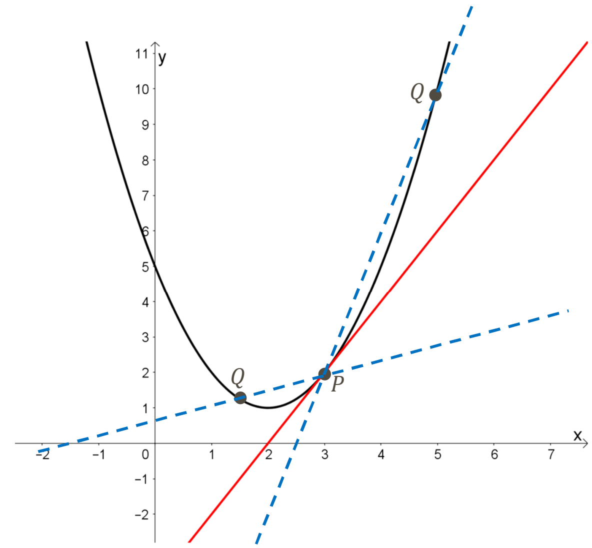


To visualize this idea, consider the parabola shown below, for which we wish to find the instantaneous rate of change at point *P*.

- To find the instantaneous rate of change at *P*, we need to find the slope of the red tangent line.
- We approximate the tangent slope by finding the slope of the dashed blue secant line through *P* and another point *Q* on the parabola.
- We gradually move *Q* closer to *P* to achieve better approximations of the tangent slope.



THE BIG IDEA



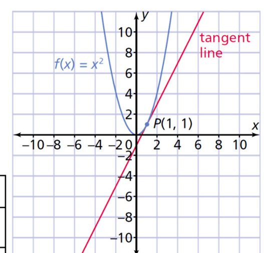
QUESTION FOR DISCUSSION Is it really necessary to approach the point of tangency from the left and the right?

To find the tangent slope (instantaneous rate of change) at the point *P* for the function $y = f(x)$:

1) Plot another point *Q* on the curve.
2) Calculate the slope of the secant line through *P* and *Q*.
3) Move point *Q* closer to *P* and calculate the secant slope again.
4) Repeat step (3) until you are certain of the value that the secant slopes are approaching.
5) Repeat the above process with the point *Q* on the other side of point *P*.
6) If the secant slopes approach the same value when *Q* gets close to *P* from both sides, use that value as the tangent slope at *P*.

Example

Determine the slope of the tangent line to the function $y = x^2$ at the point (1, 1).



For points *Q* to the left of *P*:

Q	*P*	Δy	Δx	Slope of Secant *PQ*
(0, 0)	(1, 1)	0 – 1 = –1	0 – 1 = –1	$\frac{\Delta y}{\Delta x} = \frac{-1}{-1} = 1$
(0.5, 0.25)	(1, 1)	–0.75	–0.5	1.5
(0.9, 0.81)	(1, 1)	–0.19	–0.1	1.9
(0.99, 0.9801)	(1, 1)	–0.0199	–0.01	1.99

Approaching 2

For points *Q* to the right of *P*:

Q	*P*	Δy	Δx	Slope of Secant *PQ*
(2, 4)	(1, 1)	4 – 1 = 3	2 – 1 = 1	$\frac{\Delta y}{\Delta x} = \frac{3}{1} = 3$
(1.5, 2.25)	(1, 1)	1.25	0.5	2.5
(1.1, 1.21)	(1, 1)	0.21	0.1	2.1
(1.01, 1.0201)	(1, 1)	0.0201	0.01	2.01

Approaching 2

Therefore, the slope of the tangent at the point (1, 1) is 2.

A different method…

Example

Find the instantaneous rate of change of the function $f(x) = -2\sqrt{-x+3}+5$ where $x = 2$.

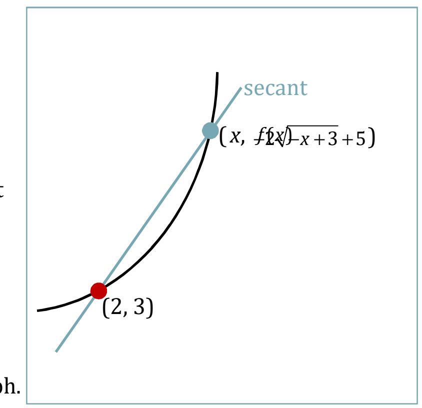
Solution

We need to find the slope of the **tangent** line at the point where $x = 2$.

Using the equation, we know that $f(2)=3$.

We'll use several secant lines to estimate the slope of the tangent.

Let's make a formula that calculates the secant slope from (2, 3) to any other point on the graph.



$$m_{\text{secant}} = \frac{\Delta f(x)}{\Delta x} \quad \begin{matrix}\rightarrow \text{rise} \\ \rightarrow \text{run}\end{matrix}$$

$$= \frac{-2\sqrt{-x+3}+5-3}{x-2}$$

$$= \frac{-2\sqrt{-x+3}+2}{x-2}$$

From the left of x=2:

x	m_{sec}
1.9	0.976176963
1.99	0.997512422
1.999	0.99975012
1.9999	0.999975

From the right of x=2:

x	m_{sec}
2.1	1.026334039
2.01	1.002512579
2.001	1.00025013
2.0001	1.000025

$\therefore$ the instantaneous rate of change where $x = 2$ is 1.